Distribution of escape times for a deterministically driven bistable system

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(Received 25 March 1999)

In this paper, we analyze the sequence of escape times for a particle in a symmetric double-well potential coupled to a chain of monodimensional oscillators and we find that, in some range of energies, the probability of escape exhibits the multimodal form that is characteristic of bistable systems driven by a periodic signal embedded in noise. We identify two different modes contributing to the overall hopping dynamics of the particle, each one having a definite dependence on the energy of the chain. Those findings suggest a model for internal fluctuations that could be useful in the study of some problems of interest in physics and biology.

PACS number(s): 05.40.−a

I. INTRODUCTION

The appearance of fluctuations in deterministic systems is a problem of general interest and has been an important subject of statistical mechanics for a long time [1]. The oldest successful derivations of equations of motion for systems interacting with a heat bath were done in the context of Brownian motion [2]. These derivations start from the Hamiltonian model of a system coupled to a properly chosen heat bath and, after elimination of the bath variables, arrive at a (generalized) Langevin equation for the system variables alone. The detailed character of the noise terms appearing in this Langevin description is dependent upon the model of heat bath and the kind of coupling that are assumed. Usually, a collection of harmonic oscillators is used to model the heat bath variables. In this case, it can be proved that, in the so-called weak coupling limit, the noisy term describing the fluctuations is Gaussian and obeys a fluctuation-dissipation theorem.

In the context of stochastic dynamics, the problem of the escape of a particle from a basin of attraction has a long and distinguished history. In particular, time-interval sequences associated with the escape of a particle from a basin of attraction have been the subject of a great deal of interest in the last few years for bistable [3,4] as well as for metastable [5,6] and excitable [7,8] systems. In the study of stochastic resonance phenomena in bistable systems, for example, the use of distributions of switching times has become instrumental to gain some understanding about the coherence in the response to periodic forcing embedded in noise [4]. On the other hand, in the context of stochastic models of neuronal behavior, the importance of bistability to explain basic characteristics of the dynamics of sensory neurons has been pointed out by Longtin and co-workers [3]. Those authors have stressed the role of noise in the transmission of sensory information by showing that, for a rather simple bistable system subject to a combination of subthreshold forcing and noise, the characteristic multimodal structure of the probability density of escape times, which cannot exist in absence of the noise term, exhibit all the substantive features of experimental interspike intervals histograms recorded from periodically forced sensory neurons.

The aim of this paper is to analyze the distribution of escape times of a bistable system embedded in a purely deterministic environment and to show that here too, a multimodal structure is obtained. In particular, we start from a fully Halmiltonian model of a particle moving in a double well potential and coupled to a chain of one-dimensional oscillators and we study the effects that on its hopping dynamics has the amount of energy made available to the whole system. By analyzing the distribution of escape times of the particle, we are able to show that despite its purely deterministic dynamics, it makes sense to describe the escape process as promoted by the action of a noisy signal coming from the environment.

In stochastic models of the well-to-well hopping of a particle in a bistable potential, a forcing term embedded in noise is usually used to drive the deterministic equation of motion for the particle. Thus, some control over the signal that is being delivered to the system is assumed when using those models [3]. At variance with the system studied by Longtin *et al.* and many others, in the model at hand we do not have control over the signal driving the tagged particle and it is only *a posteriory* that a distinction can be made between oscillating and noiselike components of the forcing. Thus, this model allows us to analyze some consequences of non-additive forcing on the dynamics of a particle in a double well potential.

II. THE MODEL

Let us consider first a one-dimensional chain of N linearly coupled oscillators described by a dimensionless Hamiltonian in the form

$$H=\sum_{k=1}^{N}\left[\frac{\dot{x}_k}{2}+\frac{1}{2}c^2(x_k-x_{k-1})^2+V(x_k)\right]. \qquad (1)$$

In the absence of an on-site potential $V(x)$, this model would describe acoustic vibrations in which c is the speed of sound in units of lattice constant. In our numerical study, we have set $c=0.5$. Here we shall also assume that the on-site potentials corresponding to the sites have the general form

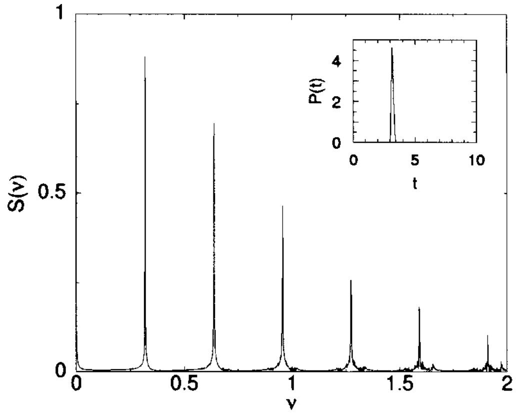

FIG. 1. Spectral density associated with the process $s_k(t)$ with arbitrary k, for a closed, one-dimensional chain of harmonic oscillators. In the inset, the first passage time distribution function is depicted, showing a narrow peak around half the natural period of the oscillators. All the magnitudes are given in arbitrary units.

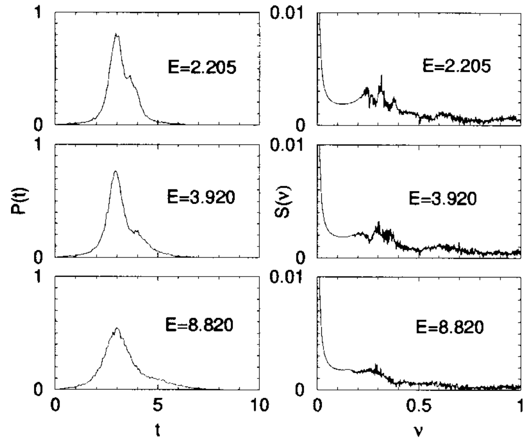

FIG. 2. Distributions of first passage times (left panels) and the corresponding spectral densities of the signal $s(t)$ as a function of the frequency, both in arbitrary units (right panels) for a given element in a closed, one-dimensional chain of soft oscillators ($\alpha=0$).

$$V(x)=\frac{1}{2}\,\frac{x^2+\alpha x^4}{1+x^2}. \tag{2}$$

The case with $\alpha=1$ corresponds to the harmonic potential with unit angular frequency ($\omega_0=1$) and those with $\alpha\neq1$ correspond to the so-called *soft potentials*. For $\alpha=0$, the potential $V(x)$ is harmonic at low amplitudes and saturates to a constant at high amplitudes. Translationally invariant, one-dimensional chains with soft on-site potentials have been the subject of some interest recently in the context of stochastic localization associated with the interplay of anharmonicity and noise [9].

To characterize the dynamics of each oscillator in the chain, we have mapped its motion into a point process given by

$$s_k(t)=\sum_i \delta(t-t_i^k), \qquad k=1,\dots,N, \tag{3}$$

where $s_k(t)$ is a dimensionless function and $\{t_i^k;i=1,2,\dots\}$ are the instants of time at which the position of the kth oscillator crosses in either direction its stable equilibrium point at $x=0$. As all the oscillators in the chain behave in a similar way, we can safely drop the subindex k in Eq. (3) and refer simply to the ''signal'' $s(t)$. From it, we can obtain the distribution of crossing times by the equilibrium point of each oscillator $P(t)$ and also the time correlation function $C(t)$, and its associated spectral density $S(\nu)$. In Fig. 1, we have plotted this last quantity for an arbitrary unit of a closed chain of $N=50$ harmonic oscillators ($\alpha=1$). As we can observe, this function shows quite narrow peaks at the frequency $\nu=\omega_0/\pi$ and its harmonics. This fact allows us to describe the chain as performing a global periodic oscillation with a frequency $\omega_0=1$. The existence of a frequency-locked motion is corroborated by the distribution of crossing times by the equilibrium point, which is shown in the inset of Fig. 1. Indeed, $P(t)$ shows a quite narrow peak around half the period of the motion in the harmonic well, $T_0/2=\pi/\omega_0$. The form of both, the spectral density and the distribution of crossing times, are independent of the initial energy made available to the whole chain and allows us to visualize the motion of each unit as performing an oscillation with a slowly modulated amplitude.

At variance with the harmonic case, the distribution of crossing times associated with a one-dimensional chain of soft oscillators is quite broad. In Fig. 2, such a distribution is depicted for an arbitrary oscillator and for some values of the energy. As we can see, it gets smoother and broader as the energy is increased. Thus, there are not a single time scale for the crossing process by the stable equilibrium point at $x=0$ and consequently, the motion of each oscillator lacks coherence. This fact is corroborated by the corresponding spectral densities which are also depicted in Fig. 2. As we can observe, these power spectra reflects basically a noisy background, this effect being more evident as the energy of the chain increases.

Let us consider now a chain of $N+1$ oscillators in which those numbered 1 to N has on-site potentials given by Eq. (2). On the other hand, the particle at the $N+1$ site is subjected to the typical bistable potential

$$V(x)=-\frac{x^2}{2}+\frac{x^4}{4}. \tag{4}$$

This function has two symmetric minima separated by a potential barrier whose height is equal to 0.25. As before, we assume the existence of periodic boundary conditions, so that the oscillators 1 and $N+1$ are coupled in the same way as any other pair of nearest neighbors. In Fig. 3, a sketch of this

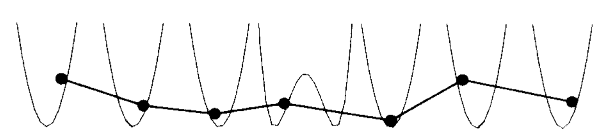

FIG. 3. Diagram sketching part of a one-dimensional chain of interacting harmonic oscillators coupled to a particle moving in a double well potential. The segments connecting neighboring particles represent interactions. The tagged particle can hop from well to well as a consequence of its coupling with the rest of the oscillators in the chain.

system is depicted in which it is apparent that the tagged particle (the oscillator moving in the bistable potential) can jump from well to well if it has enough energy to surmount the potential barrier between them, the probability of this event being dependent on the amount of energy locally delivered to it by the rest of the system.

When a chain made of identical units is coupled to a particle in a double well potential, the behavior of the different oscillators in it becomes dependent on the distance to the tagged particle. Nevertheless, this dependence decays with the distance and for a chain made of a big number of single oscillators, the vast majority of them behaves in an analogous way which is independent on whether the tagged particle is present or not. So, we can characterize the asymptotic behavior of the chain by using an oscillator which is far away from the tagged particle. We have taken a closed chain of $N=49$ elements and the oscillator characterizing the behavior of the chain is just the opposite to the tagged particle.

The dynamics of the tagged particle can also be characterized by mapping its motion into the point process

$$r(t)=\sum_i \delta(t-t_i), \tag{5}$$

where $\{t_i;i=1,2,\dots\}$ are now the instants of time at which its position crosses in either direction the unstable equilibrium point at $x=0$. In this form, a dimensionless signal with the structure of a Dirac comb is constructed which is analogous to the one employed in theoretical neurobiology to model the spike train produced by a neuron [10].

III. THE DISTRIBUTION OF FIRST PASSAGE TIMES

In this section, we shall analyze some aspects of the dynamics of the tagged particle by using the information carried by the signal $r(t)$. In order to obtain a realization of this signal, we must solve the equations of motion of all the particles in our system to obtain the position of the tagged particle as a function of time. We have done this part of the calculation by means of a fourth order Runge-Kutta algorithm with time step $h=0.001$, and using different sets of initial conditions. Initially, all the oscillators in the chain are placed at their equilibrium points and their velocities are distributed around the central value by using a Gaussian with a very narrow dispersion. The tagged particle is placed at rest at the bottom of the left well.

Each one of those initial conditions allows us to calculate the total amount of energy that is delivered initially to the whole chain. If that energy is great enough, the tagged particle will occasionally jump from well to well as time goes by. We have obtained 300 realizations of the position of the tagged particle as a function of time, each one consisting of 2^{21} time steps following a initial evolution of 2^{17} time steps that we have discarded for calculation purposes. During its temporal evolution the tagged particle performs very many hops from well to well, and this fact allows us to determine the statistics of the crossing times by the top of the barrier. The total number of jumps performed is dependent on the energy delivered initially to the whole chain. In our simulation, this number ranges from 7×10^4 ($E=17.81$) to 3×10^3 ($E=2.16$). The distribution of escape times has been constructed by computing the time intervals between successive crossings and accumulating them to the bins of a suitable discretization of the time axis. On the other hand, we have calculated the correlation function of the signal $r(t)$ and its associated spectral density.

It is well known that the behavior of a system of coupled oscillators is strongly dependent of the coupling strength. For a weak enough coupling ($c\leqslant0.3$), the behavior of the tagged particle is almost independent of the motion of the chain, provided that the energy is not very great with respect to the barrier height of the bistable potential. In this regime, the tagged particle can remain for a long time in one of the wells until a fluctuation in its energy allows it to surmount the barrier and to hop to the other well. Moreover, if such a process takes place, the weak coupling makes very improbable the losing of the energy by the tagged particle and so, it will remain for a long time jumping from well to well with its natural frequency.

For strong coupling ($c\geqslant0.7$), the dynamics of the whole system is governed by the frequent interchange of energy between the tagged particle and the chain of oscillators. If the chain has enough energy to allow the tagged particle to jump from well to well, the distribution of escape times develops a broad monomodal estructure centered at the natural frequency of the oscillators in the chain. In this work we are interested in the intermediate coupling regime ($0.3<c<0.7$), where the appearance of a complex pattern of forcing is to be expected. The results presented here corresponds to the case $c=0.5$.

A. Harmonic chain

We can consider the oscillatory motion of the chain as providing the driving on the tagged particle although we can no longer speak of a forcing signal, as would be the case if we had an external signal acting on the particle. To describe the nature of the driving in our case it is necessary to describe the motion it gives rise, determining whether it has a smooth component or not. To start, we have studied the distribution of escape times from a well, because this quantity provides useful information about the existence of oscillatory components in the overall motion. In Fig. 4, the distribution of escape times, $P(t)$ is depicted for a the chain consisting of $N=49$ harmonic oscillators of unit frequency. When the average energy per oscillator in the chain is far greater than the corresponding to the potential barrier of the tagged particle ($E=17.81$ means an average energy per oscillator of 0.3562), the motion in the double well is dominated by jumps associated with the natural motion of the chain and $P(t)$ is clearly monomodal with the peak centered around $T_0/2$. In fact, for this range of energies, the existence of a barrier does not affect appreciably the dynamics of the tagged particle and its crossings by the top of the barrier cannot be associated with a regime in which clearly defined dwelling times in each well are separated by sudden well-to-well jumps. In Fig. 5, the height of this very first peak of $P(t)$ has been plotted to show its nonmonotonous dependence on the chain energy. In fact, superposed to an overall decreasing trend, we can observe some energies for which the contribution to the motion of the tagged particle of this peak is enhanced.

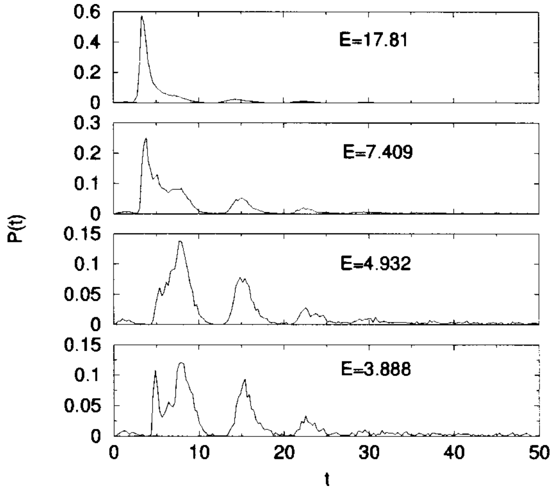

FIG. 4. Distribution of escape times for the tagged particle at some values of the adimensionalized energy. Each one of these distributions has been obtained by collecting a big number of intervals between the successive crossing of the particle over the top of the barrier and arranging them to form a histogram.

On the other hand, when each oscillator in the chain has an energy that, on the average, is lower than the barrier that the tagged particle must surmount to go to the other well, the distribution of escape times develops a multimodal form associated with a fundamental period and its higher harmonics. That means that the motion of the tagged particle in this range of energies can be described as a combination of residence times in each well followed by rapid escape processes driven by local excitations. This multimodal component of $P(t)$ is dominant for very low energies but, as shown in Fig. 4, for an average energy per oscillator equal to 0.148 ($E = 7.409$), the dynamics of the tagged particle clearly mixes both components. Observe that the second peak of the multimodal structure always remains quite small, its dominance comimg from the relative lowering of the first peak. This fact is also evident by looking at Fig. 6, in which a closer look at the evolution of $P(t)$ is taken. There, it can be observed that

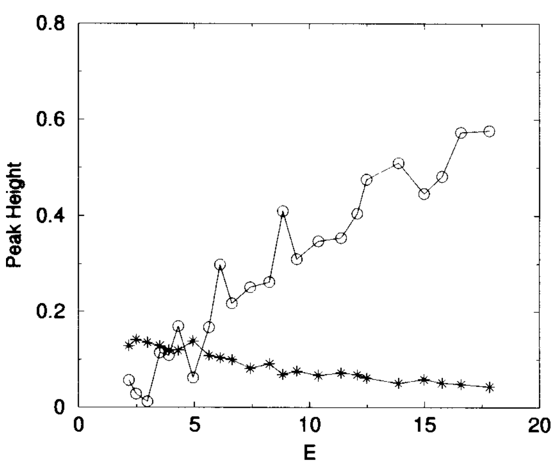

FIG. 5. Height of the first two peaks of the distribution of escape times for the tagged particle as a function of the total energy of the chain. Open circles indicate the height of the first peak and asterisks correspond to the second peak. Lines are depicted as a guide to the eye.

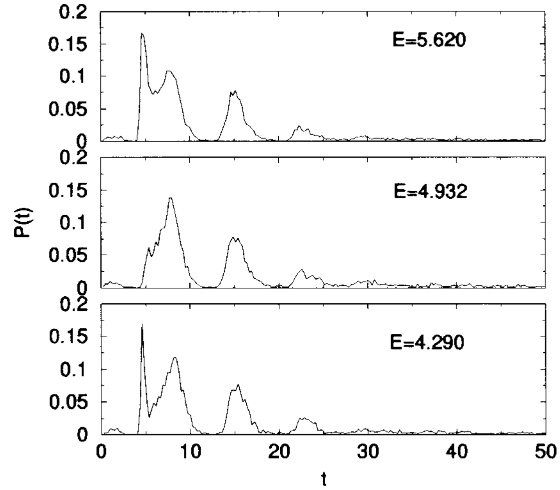

FIG. 6. Probability of escape times for some values of the energy around $E = 5$. For $E = 5.620$, the peak at time $T_0/2$ is clearly dominant with respect to the first peak of the multimodal structure associated with jumps between wells. This is also the case for $E = 4.290$. However, for an intermediate energy as $E = 4.932$, the first peak of $P(t)$ is lower than the second one.

for some energies, the contribution of the first peak suddenly decreases in relation with the dominant peak of the multimodal structure.

In the study of the motion of a particle in a bistable potential driven by a combination of a sinusoidal signal plus a noise, the existence of a multimodal $P(t)$ indicates the imperfect phase locking of the interwell hopping motion to the frequency of the forcing term. This fact is usually associated with the noise-induced skipping of some jumps [3]. Here too, the multimodal form of $P(t)$ suggest that, despite being of a purely deterministic nature, we can interpret the driving in our system as providing an oscillatory component that rocks the tagged particle within each well and enhances periodically the probability of a jump over the barrier. Once this oscillatory component has driven the tagged particle to the most favorable state to jump to the other well, the action of some other components in the driving can avoid the actual completion of the jump. Thus, there is a probability that the particle skips one or more periods of the driving signal, this fact being associated with the secondary peaks of decreasing magnitude in the distribution of escape times. This skipping suggest the existence of a noiselike component in the signal as experimented by the tagged particle.

The structure of the distribution of escape times for the tagged particle can be interpreted as indicating the presence in the signal of two components with different dynamical roles. The first of those components reflects the pumping of energy in the bistable with a frequency associated with that of the harmonic oscillators. In fact, for high enough energy available in the chain, the only peak in $P(t)$ is located at the natural frequency of the harmonic oscillators in the chain. For lower energies, however, this frequency shifts to lower values. Although the overall trend of this component is correlated with the energy in the chain, for some energies, it becomes nearly inexistent. There is a second component arising from the well-to-well dynamics that mimics the behavior of a bistable under the action of a periodic signal plus a noise term. As was pointed out above, both components are not

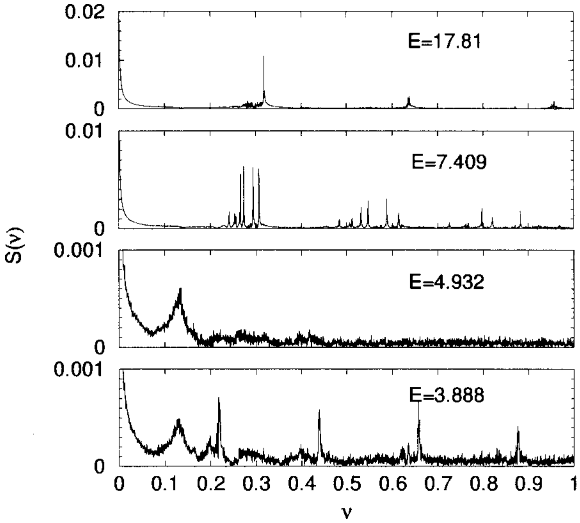

FIG. 7. Spectral density of the signal $r(t)$ as a function of the frequency (both in arbitrary units). The cases corresponding to those depicted in Fig. 4 have been plotted for comparison purposes. Observe the change of scale in the vertical axis.

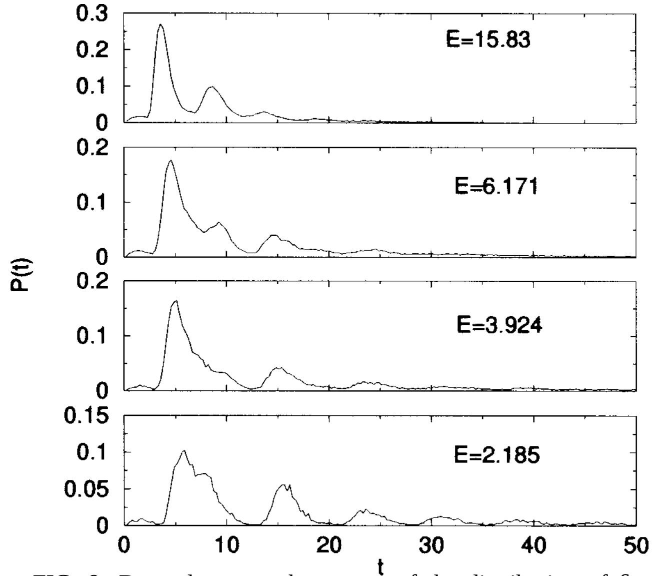

FIG. 8. Dependence on the energy of the distribution of first passage times of a particle in a double well potential interacting with a one-dimensional chain of soft oscillators ($\alpha=0$).

given as separate entities by the model. Instead, they appear mixed in a signal that comes from the dynamics of the whole chain.

Another way of characterizing the presence of different components in the signal follows from the calculation of the spectral density of the signal $r(t)$. The use of this quantity as characterizing the response of a bistable system has been very much employed in the context of stochastic resonance [11]. It is a simple task to evaluate numerically the correlation function associated to $r(t)$ [10]. From it, the corresponding spectral density $S(\nu)$ can be obtained straighforwardly by using a numerical Fourier transform. In Fig. 7, the power spectrum is depicted for the same cases presented in Fig. 4. Again, the existence of a periodic component in the signal is evidenced by the peaked structure of $S(\nu)$. On the other hand, the appearance of a continuous background in this function can be associated with the presence of a noiselike component in $r(t)$. For high energies, the power spectrum shows only the peak corresponding to T_0 (and some small harmonics). This is the hopping regime associated with a purely periodic driving in which the barrier height of the bistable potential is irrelevant. As the energy is lowered, the motion of the tagged particle becomes more involved and for $E\approx4.9$, those peaks disappear at all. At the same time, a new feature appears in the spectrum in the form of a broad peak indicating the dominance of the well-to-well motion. At lowest energies, these two mechanisms coexist.

B. Anharmonic (soft) chain

As before, the coupling of the chain to a particle in a double well potential does not affect significantly to those oscillators which are far enough from it. In order to analyze the influence of the chain on the dynamics of the tagged particle as described by the signal $s(t)$, we have depicted in Fig. 8 the behavior of its distribution of first passage times for some values of the energy available in the whole chain. At low energies, the function $P(t)$ seems to be closer to the corresponding to the harmonic chain although it lacks the contribution of the first component. In this regime, as was the case with a chain of harmonic oscillators, the well-to-well dynamics of the tagged particle is dominant and the quasiharmonic nature of the soft potential near the equilibrium point makes things very similar to the purely harmonic case. At higher energies, the multimodal structure remains but barrier crossings can be observed at any time. Observe that the time of the second and successive peaks for this range of energies are not multiples of the time corresponding to the first peak. Indeed, when the energy is increased from $E=3.924$ to $E=6.171$, a new second peak appears on $P(t)$, thus giving rise to another kind of temporal structure.

IV. CONCLUSIONS

In this paper we have studied the effects of the deterministic dynamics of a chain of coupled oscillators on the behavior of a particle in a double-well potential. The main goal was to infer some characteristics of the forcing by analyzing the motion of the tagged particle. For the case of an harmonic chain, the analysis of some quantities associated with the well-to-well hopping has lead us to distinguish two separate contributions to the forcing of the tagged particle, one describing a periodic signal associated with the natural oscillation frequency of the chain oscillators and the other being characterized as the superposition of a periodic signal and some kind of ''internal noise.'' Both components display very different behaviors when the energy of the chain is varied, one of them being dominant in the low energy range and the other in the high energy region. In particular, for some values of this control parameter, resonances in the interwell dynamics of the tagged particle have been found, indicating that for some specified energies, one of those components is enhanced.

The comparison between the two different systems we have analyzed, gives us some clues about how the coherence of the well-to-well hopping of the tagged particle is affected by the chain motion. When the oscillation modes in the chain are narrowly distributed in frequency, the structure of the signal $s(t)$ reflects this coherence. This is specially relevant when there are high energy modes in the chain. On the other hand, for some energies in low energy regime the temporal structure of the crossing times is quite independent, at least qualitatively, of the detailed form of the on-site potentials. In those cases, the hopping dynamics of the tagged particle is similar to the corresponding to a periodic forcing of subthreshold intensity embedded in noise. Nevertheless, this similarity does not hold for the full low-energy range because some effects of the coherence in the chain motion are detectable even for those energies.

The role of internal noise in neuronal dynamics and its possible consequences to sensory transduction have been the subject of some interest recently. The interplay of intrinsic oscillations and noise seems to be at the basis of the remarkable encoding properties of some sensory detectors [12]. In many stochastic models of excitable systems describing neuronal dynamics, a stochastic term describing the neural noise is added to a coherent signal in order to analyze the effects of intrinsic and/or extrinsic fluctuations on the neuron performance.

However, it is not clear at all that the oscillatory and noisy components of the membrane behavior could always be treated as separate processes. The use of a model as the one developed here, for example, in situations where the tagged particle is subjected to external periodic forcing, can result in gaining some understanding on the transduction properties of systems in which this separation is unfair.

ACKNOWLEDGMENT

The authors acknowledge the Dirección General de Investigación Científica y Técnica (DGICYT) of Spain (Project No. PB95-0536) for support.

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